MA 234, Spring 2015

Assignment 4

This homework is due Friday Feb 20.

There are total 45 points in this assignment. 40 points is considered 100%. If you go over 40 points, you will get over 100% for this homework (but not over 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should exhibit your work and contain full proofs. Bare answers will not earn you much.

This assignment covers Sections 2.2–2.4 of Textbook.

(1) [10pt] Find the images of the mapping $w = z^2$ in each case, and sketch the mapping.
   (a) The horizontal line $\{(x, y) : y = 1\}$.
   (b) The vertical line $\{(x, y) : x = 2\}$.
   (c) The rectangle $\{(x, y) : 0 < x < 2,\ 0 < y < 1\}$.
   (d) The rectangle $\{(x, y) : -1 < x < 2,\ -2 < y < 1\}$.
   (e) The infinite strip $\{(x, y) : 1 < x < 2\}$.

(2) [10pt] Find and sketch the images of the following sets under the mapping $w = z^{\frac{1}{2}}$, the principal square root function.
   (a) $\{re^{i\theta} : r > 1,\ \frac{\pi}{3} < \theta < \pi\}$.
   (b) $\{re^{i\theta} : 1 < r < 9,\ 0 < \theta < \frac{2\pi}{3}\}$.
   (c) $\{re^{i\theta} : r < 4,\ -\pi < \theta < \frac{\pi}{2}\}$.
   (d) The horizontal line $\{(x, y) : y = 2\}$.
   (e) The vertical line $\{(x, y) : x = 4\}$ (*Hint:* $u^2 - v^2 = A$ (where $A \neq 0$ is an equation of a hyperbola with asymptotes $u = \pm v$).

(3) [5pt] Find the following limits using basic arithmetic properties of limits.
   (a) $\lim_{z\to 1} \frac{z^2+1}{z+1}$.
   (b) $\lim_{z\to 1} \frac{z^2-1}{z-1}$.
   (c) $\lim_{z\to i} \frac{z^4-1}{z-i}$.
   (d) $\lim_{z\to 1+i} \frac{z^2+z-2+i}{z^2-2z+1}$.
   (e) $\lim_{z\to 1+i} \frac{z^2+z-1-3i}{z^2-2z+2}$. (*Hint:* Factor[1] both quadratic polynomials as $(z - z_1)(z - z_2)$.)

(4) [5pt] Let $f(z) = \frac{z^2}{|z|^2} = \frac{x^2-y^2+i2xy}{x^2+y^2}$.
   (a) Find $\lim_{z\to 0} f(z)$ as z approaches 0 along the line $y = x$.
   (b) Find $\lim_{z\to 0} f(z)$ as z approaches 0 along the line $y = 2x$.
   (c) Find $\lim_{z\to 0} f(z)$ as z approaches 0 along the parabola $y = x^2$.
   (d) What can you conclude about the limit of $f(z)$ as $z \to 0$?

— see next page —

[1]The following may help. (1) If z_1, z_2 are roots of $z^2 - bz + c$, then $z^2 - bz + c = (z - z_1)(z - z_2)$. (2) If you know one root, you can find the other by observing that $z_1 + z_2 = b$.

(5) [5pt] Does $\lim_{z \to -4} \operatorname{Arg}(z)$ exist? Why? (*Hint:* Use polar coordinates and approach -4 from the upper and lower half-planes.)

(6) [5pt] Let $f_1(z)$ be the principal square root function and $f_2(z)$ be the complementing branch of square root, $f_2(z) = -f_1(z)$. Use polar coordinates to find and sketch image of
   (a) quadrant II ($x < 0$, $y > 0$) under the mapping $w = f_1(z)$,
   (b) quadrant II ($x < 0$, $y > 0$) under the mapping $w = f_2(z)$,
   (c) the right half-plane $\operatorname{Re}(z) > 0$ under the mapping $w = f_1(z)$,
   (d) the right half-plane $\operatorname{Re}(z) > 0$ under the mapping $w = f_2(z)$.

(7) [5pt] Describe and sketch Riemann surface for $z^{\frac{1}{3}}$. (What sheets does it consist of? How are they attached to each other?).